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Directed percolation in 2+1 dimensions

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Abstract. We present results of Monte Carlo simulations for directed percolation in 2+1 dimensions very close to the percolation threshold. Our values for p_c for bond and site percolation on the BCC and simple cubic lattices are more precise than previous estimates by one to two orders of magnitude. The improvement on the critical exponents describing the behaviour at p_c is less impressive, but they are also substantially more precise than previous ones.

1. Introduction

Directed percolation [1-4] can be understood as a prototypical model for the spreading of some influence (such as an epidemic, a forest fire, or a political opinion) under a bias pushing the spread predominantly in one direction. In this interpretation, the preferred direction is a spatial direction, and directed percolation in $d+1$ dimensions is relevant for the spreading in $D=d+1$ dimensions of space.

A more natural and more interesting interpretation arises, however, if we take the biased direction as the direction of time. In this case, directed percolation describes the—causal, but otherwise undirected—spreading in d dimensions of space, of an influence which does not involve a conserved quantity and which does not exhaust its resources. In this context it is sometimes called the simple epidemic with recovery (without immunisation), or the basic contact process [5]. Its critical properties near the threshold for infinite spreading should be also those of Schlögl's first model [6] or of the Malthus-Verhulst model with spatially inhomogeneous fluctuations.

Within this latter interpretation, it was found in [7] (see also [8-10] that the critical behaviour of directed percolation is the same as that of reggeon field theory (RFT) [11-13], a theory studied intensively by particle physicists in the 1970s. Since the physical number of dimensions for RFT was 2+1, it was also for this dimensionality that most intensive studies were performed. The methods used were mostly high-temperature expansions [12], loop expansions, and ε expansions around 4+1 dimensions [13].

Directed percolation in 1+1 dimensions has been studied very intensely during recent years [14], with the result that the RFT results are now obsolete. In contrast, and in spite of recent work done within the directed percolation interpretation proper [15-18], it seems that the RFT exponents and amplitude ratios still represent the most

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precise overall description for the critical behaviour of directed percolation in $2+1$ dimensions.

It is the aim of the present paper to change this situation by means of Monte Carlo simulations. These simulations are very straightforward. We work on discrete spatial lattices, and we use discrete time. At each time step, we keep a list of growth sites, representing the cluster of sites wetted (or infected) at the previous time step, and we build up a new list of sites, neighbouring the old growth sites, which are to be the growth sites of the next generation. We start with a single growth site at time zero. In order to simulate bond percolation, we let every growth site infect each neighbour independently with probability p. For site percolation, we assume that a growth site infects either none of its neighbours or all of them. Multiple infection, leading to a site appearing twice in the list of growth sites, is avoided by keeping a flag at each site. This flag is on if the site has already been infected in the present time step, and it is cleared before going on to the next time step. It is only the clearing of these flags (corresponding to recovery without immunisation) which makes directed percolation in $d+1$ directions different from the spreading of undirected percolation in d directions [19, 20].

2. The simulations

Spreading of the contact process on a square lattice corresponds, when seen in $2+1$ dimensions, to percolation on a BCC lattice with the preferred direction along one of the axes [21]. On the other hand, spreading in a simple cubic lattice with bias along the space diagonal leads, when projected down to two dimensions, to a triangular lattice with spreading only along three of the six possible directions (the three other directions would correspond to spreading backward in time).

I made extensive simulations only very close to the critical point, for bond percolation on both lattices. For each value of p, between 10^5 and 2.5×10^5 clusters were simulated. The number of time steps per cluster was 1000, unless the cluster had already died out earlier. Somewhat less extensive simulations were also made for site percolation on both lattices, also for p very close the p_c. The total CPU time used for all these simulations was about 70 hours on a Sun 4 workstation.

The measured quantities were the survival probability $P(t)$, giving the chance that after t time steps there is still at least one growth site, the average square radius $R^2(t)$, and the average number of growth sites $n(t)$.

At the critical point $p=p_c$, we expect scaling laws

$$P(t)\sim t^{-\delta} \tag{1}$$

$$n(t)\sim t^{\eta} \tag{2}$$

$$R^2(t)\sim t^{z}. \tag{3}$$

Notice that $n(t)$ is the average taken over *all* clusters, including those which have already died out, while $R^2(t)$ is of course averaged only over the growth sites in the surviving clusters. The average number of growth sites per surviving clusters is $N(t)=n(t)/P(t)$. The fractal dimension d_f of the surviving clusters at *fixed time* is defined as $N\sim R^{d_f}$, which gives $d_f=2(\eta+\delta)/z$.

The density of growth sites at the origin $\boldsymbol{r}=0$ can be bounded in terms of $P(t)$. The probability of having a growth site at $\boldsymbol{r}=0$ is indeed equal to the probability

$P(t/2)$ of having at least one growth site a time distance $t/2$ away from the seed, multiplied by the same probability of having at least one growth site the same distance away from the point $(0, t)$ times the probability that some site is common to both sets of growth sites. Omitting the last probability, we have the inequality

$$\rho(\boldsymbol{r}=0, t) \leqslant [P(t/2)]^2. \tag{4}$$

If the r dependence of the density of growth sites scales with the exponent z

$$\rho(\boldsymbol{r}, t) \simeq \rho(0, t) F(r^2/t^z) \tag{5}$$

then this implies $\rho(0, t) \sim t^{\eta - zd/2}$ and

$$\tfrac{1}{2}dz - \eta \geqslant 2\delta. \tag{6}$$

For $p > p_c$, it is clear that the inequality (4) becomes saturated for $t \to \infty$, and using a scaling ansatz for general p near p_c one sees easily that the 'hyperscaling' relation (6) should become an equality too [9]. It gives $d_f = d/2 + \eta/z$.

Results for the three quantities $P(t)$, $n(t)$ and $R^2(t)$ obtained for bond percolation on the SC lattice are shown in figure 1. It is obvious from this figure that $0.3821 < p_c < 0.3824$, which is already more precise than the best previous estimates, $p_c = 0.3825 \pm 0.001$ [18] and $p_c = 0.382 \pm 0.001$ [15]. Very similar curves are obtained for the BCC lattice and for site percolation. In order to obtain more precise estimates for p_c and for the critical exponents, we plot in figure 2 the slopes of the curves shown in

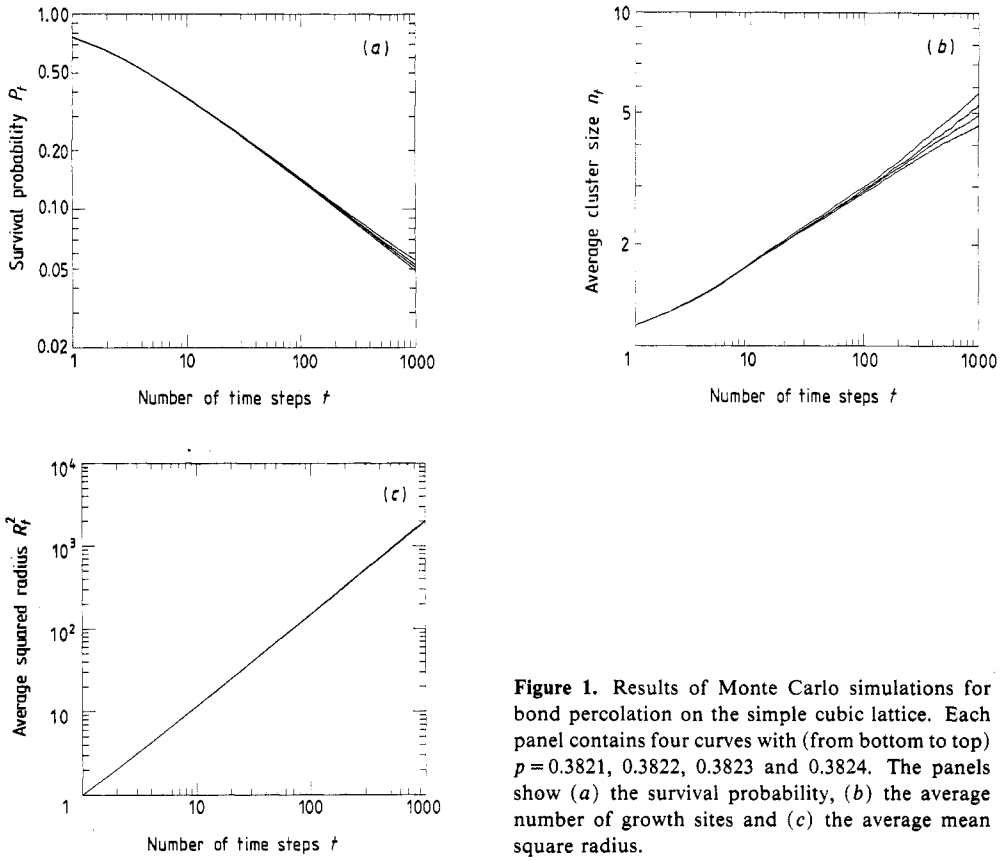

Figure 1. Results of Monte Carlo simulations for bond percolation on the simple cubic lattice. Each panel contains four curves with (from bottom to top) $p = 0.3821$, 0.3822, 0.3823 and 0.3824. The panels show (*a*) the survival probability, (*b*) the average number of growth sites and (*c*) the average mean square radius.

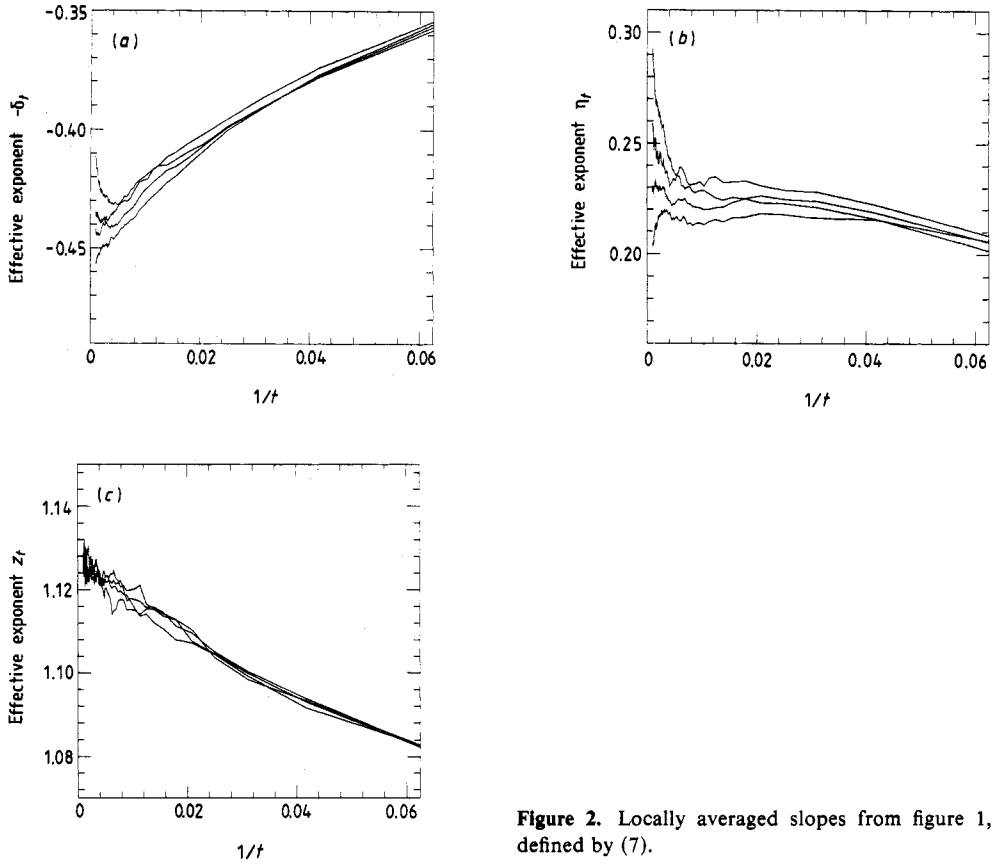

Figure 2. Locally averaged slopes from figure 1, defined by (7).

figure 1, and in figure 3 the analogous results for the BCC lattice. In figures 4 and 5, we show the average number of growth sites $n(t)$ for the two site percolation problems.

More precisely, the quantities shown in figures 2 and 3 are averaged local slopes

$$\delta_t = \frac{\log[(P(t)/P(t/8)]}{\log 8} \tag{7}$$

and similarly for η_t and z_t.

In general we have to expect corrections to scaling of the type

$$P(t) \propto t^{-\delta}\left(1+\frac{a}{t}+\frac{b}{t^{\delta'}}+\ldots\right) \tag{8}$$

and similarly for the other two observables with correction-to-scaling exponents η', respectively z', instead of δ'. This implies for the local slope δ_t defined in (7) the behaviour

$$\delta_t = \delta+\frac{a}{t}+\frac{\delta' b}{t^{\delta'}}+\ldots \tag{9}$$

and analogous expressions for η_t and z_t.

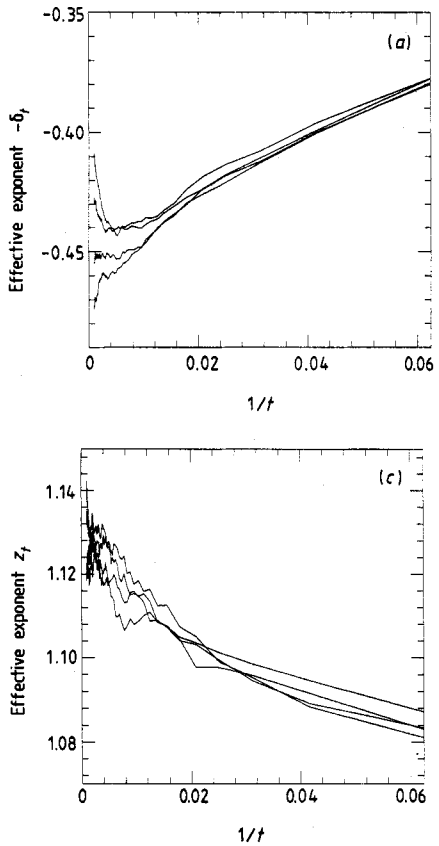

Figure 3. Same as figure 2, but for the BCC bond percolation. The four curves in each panel correspond (from bottom to top) to $p = 0.2872$, 0.2873, 0.2874 and 0.2875.

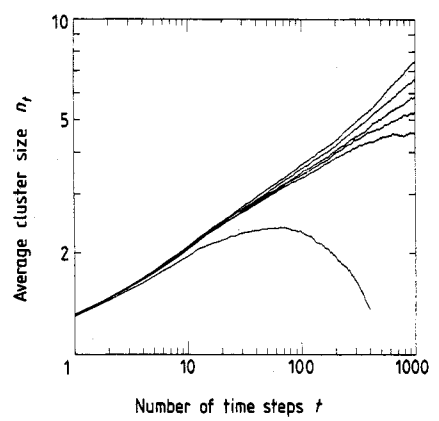

Figure 4. Same as figure 1(*b*), but for SC site percolation. The six curves correspond (from bottom to top) to $p = 0.4315$ (one of the two alternatives obtained in [15]), 0.4349, 0.4351, 0.4353, 0.4355 and 0.4357.

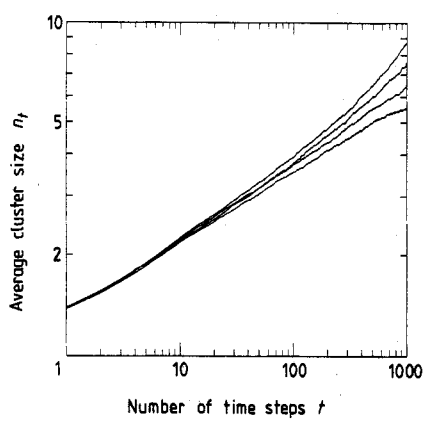

Figure 5. Same as figure 4, but for BCC site percolation. The four curves correspond (from bottom to top) to $p = 0.3444$, 0.3446, 0.3448 and 0.345.

Since we have plotted the slopes in figures 2 and 3 over $1/t$, any deviation from straight lines of the curves corresponding to $p = p_c$ indicates a correction-to-scaling exponent less than 1. We see that there are indeed clear indications for such corrections. They are most prominent for $P(t)$ in the case of the SC lattice (for both the bond and site problems), and for $R^2(t)$ in case of the BCC bond problem. In these cases, the correction-to-scaling exponents are close to $\frac{1}{2}$, $\delta'^{(\mathrm{SC,bond})} \simeq \delta'^{(\mathrm{SC,site})} \simeq z'^{(\mathrm{BCC,bond})} \simeq 0.5$. In contrast, our best fits gave $\delta'^{(\mathrm{BCC,bond})} \simeq z'^{(\mathrm{SC,bond})} \simeq 0.7$, and $\eta'^{(\mathrm{BCC,bond})} \simeq \eta'^{(\mathrm{SC,bond})} \simeq 0.6$. Since the correction-to-scaling exponents should be universal (i.e. independent of the lattice), we conclude that indeed all of them are $\simeq 0.6$. The corrections to scaling for the site problems are consistent with this but less significant, due to the less extensive simulations.

Using this, our best final estimates for p_c and for the critical exponents become

$$p_c^{(\mathrm{BCC,bond})} = 0.28730 \pm 0.00006 \tag{10}$$

$$p_c^{(\mathrm{SC,bond})} = 0.38216 \pm 0.00006 \tag{11}$$

$$p_c^{(\mathrm{BCC,site})} = 0.3445 \pm 0.00012 \tag{12}$$

$$p_c^{(\mathrm{SC,site})} = 0.43525 \pm 0.00013 \tag{13}$$

and

$$\begin{aligned} z &= 1.134 \pm 0.004 \qquad & \delta &= 0.460 \pm 0.006 \\ \eta &= 0.214 \pm 0.008 \qquad & d_f &= 1.187 \pm 0.007. \end{aligned} \tag{14}$$

In these estimates, we have put most weight on those data which show least corrections to scaling, and we have taken into account the hyperscaling relation (6).

3. Discussion

For all four lattices, the percolation threshold p_c is in reasonable agreement with previous estimates [15, 17, 18], but is more precise by more than one order of magnitude (in particular, the ambiguity in $p_c^{(\mathrm{SC,site})}$ found in [15] is resolved in favour of the bigger value). This is mainly because the previous results were based on exact enumerations which involved much shorter times ($t < 20$ instead of $t = 1000$ in the present paper).

Since the noise inherent in Monte Carlo data prevents a clean correction-to-scaling analysis, the improvements on the critical exponents are much less dramatic. Nevertheless, our exponents seem more precise than those given previously. The overall agreement is reasonable. The biggest discrepancy with the RFT estimates of [12] is for the exponent z which was obtained there as $z = 1.16 \pm 0.01$. On the other hand, the present value of z agrees with that of [13], while the value $\eta = 0.26 \pm 0.02$ given there is somewhat higher than ours. Our value of η seems indeed to be the lowest published so far.

Most recent estimates of critical exponents do not assume the validity of the hyperscaling relation (6) and quote their results in terms of differently defined exponents. For instance, the exponent γ is defined via the cluster size in $d+1$ dimensions, i.e. via the total number S of infected spacetime points, as $S \sim (p_c - p)^{-\gamma}$ for $p \nearrow p_c$, and it satisfies the scaling relation $\gamma = \nu_\parallel(1+\eta)$. The values quoted in [18] are $\gamma = 1.60 \pm 0.04$, but $\nu_\parallel(1+\eta) = 1.61 \pm 0.02$. This makes a direct comparison with the present results somewhat difficult. Also, I have not attempted to estimate any exponent describing the behaviour off the critical point, such as the transverse and longitudinal correlation length exponents $\nu_\perp$ and $\nu_\parallel$.

A more interesting comparison with previous results can be made by combining our values of p_c with the previous exponent estimates. As remarked repeatedly [4, 15, 16, 18], the largest uncertainty in estimates of critical exponents from exact series is due to the uncertainty of p_c. For instance, the error in the estimate $\gamma = 1.57 \pm 0.04$ of [15] is nearly exclusively due to the error in p_c, as seen from their p_c-dependent estimates: $\gamma = 1.565 \pm 0.003 + 37\Delta p_c$ with $\Delta p_c = p_c - 0.382$ for the SC bond problem, $\gamma = 1.494 \pm 0.001 + 22\Delta p_c$ with $\Delta p_c = p_c - 0.4315$ for the SC site problem, $\gamma = 1.570 \pm 0.004 + 28\Delta p_c$ with $\Delta p_c = p_c - 0.344$ for the BCC site problem, and $\gamma = 1.59 \pm 0.002 + 36\Delta p_c$ with $\Delta p_c = p_c - 0.288$ for the BCC bond problem. Inserting here our values of p_c, we obtain a consistent average $\gamma = 1.571 \pm 0.006$, i.e. an error reduction by nearly one order of magnitude. Similarly, we get from [15] the improved estimates $\nu_\parallel = 1.286 \pm 0.005$ and $\nu_\perp = 0.729 \pm 0.008$. From these we obtain $z = 2\nu_\perp/\nu_\parallel = 1.134 \pm 0.013$ and $\eta = \gamma/\nu_\parallel - 1 = 0.222 \pm 0.006$. Both values are in excellent agreement with those given in (14). In the same way, the estimate of the resistivity susceptibility exponent γ_R of [16] improves from 2.70 ± 0.04 to $\gamma_R = 2.703 \pm 0.006$. Finally, using $p_c = 0.382\,16$ for the SC bond problem brings the estimate of γ of [18] down to 1.58 ± 0.02, improving thus the agreement with [15].

In conclusion, we can say that our improved values of the percolation thresholds represent an important result of the present paper. Combined with previous results from series expansions, they give estimates of the critical exponents which are internally consistent. They are also consistent with the estimates of the exponents obtained directly from our Monte Carlo data, and typically have the same errors.

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